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Change of Probability Measure in Weighted Empirical Cumulative Distribution Function

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ABSTRACT

Change of probability measure is a fundamental and intensely treated concept in measure theory, and it has important implications and applications in probability theory, financial modeling and engineering and the theory of stochastic processes. Indeed, the capability of changing the probability measure that defines a probability space can be handy in any situation in which a risk measure should be estimated, and variance reduction or importance sampling techniques are used to reduce the computational cost of risk function estimation. In these cases, the data samples available to estimate the risk function value are not computed according to the actual statistical distribution that models the uncertain input parameters, and a change of probability measure should be introduced to recover the correct estimation of the quantity of interest.

The approach here illustrated for changing the probability measure derives directly from the methodology illustrated in [Amaral et al., 2016], and it relies on the introduction of a Weighted Empirical Cumulative Distribution Function (WECDF). The main difference is in the approach used to change the weights, that relies on empirical density function ratios rather than cumulative distributions and, consequently, in the numerical algorithms used to compute the weights.

The development of new efficient approaches to robust or reliability-based optimization problems requires the ability to obtain reasonable estimates of the risk function during the optimization process even when the samples are few and have not been generated according to the actual distribution of the input random variables. In most cases only a small finite number of evaluations obtained by discrete sampling of the input parameters is available, and this can be a problem if a good estimation of the risk measure governing the problem is required. The methods commonly used in these cases either rely on response surfaces [Crestaux et al., 2009], or sophisticated quadrature laws [Witteveen and Iaccarino, 2012, Congedo et al., 2013], or, finally, on multi-level sampling methods [Giles, 2015]. Appropriate use of importance sampling and, consequently, of change of probability measure, alone or in conjunction with the previously

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mentioned methods, could help to drastically reduce the number of costly evaluations of the objective function required.

The change of probability measure is obtained through a constrained optimization process aimed to assign suitable values to the weights of the WECDF:

$$\begin{aligned} \mathbf{w} = \arg\min_{\mathbf{w}} \quad & \omega^2(\mathbf{w}) \\ \text{s. to:} \quad & w_i \geq 0,\ i = 1, \ldots, n \\ & \sum_{i=1}^{n} w_i = 1 \end{aligned} \tag{1}$$

with ω^2 a suitable objective function that will be described in the full work. The problem can be solved using a non-negative least squares algorithm, like the Lawson and Hanson algorithm [Lawson and Hanson, 1995], or transforming it into a convex quadratic programming problem [Amaral, 2015]. Figure 1 reports the result obtained to restore the proper CDF related to the function $f = \frac{1}{n} \sum_{i=1}^{n} (2\pi - u_i) \cos(u_i - d_i)$ with $\mathbf{u} \in [0, 3]^n, \mathbf{d} \in [0, 2\pi]^n$ and $n = 6$. The probability measure was changed to let the random vector $\mathbf{u}$ have components with a uniform distribution function.

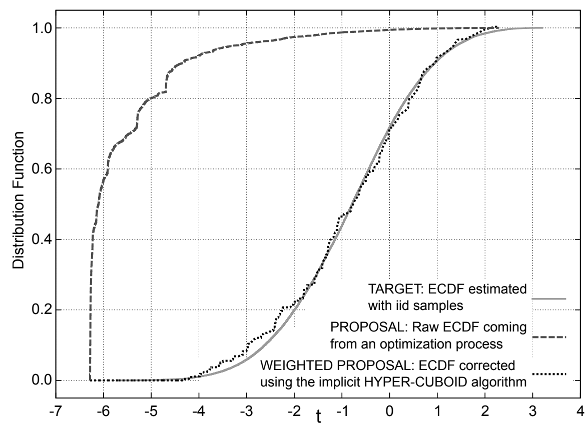

Figure 1: *Change of probability measure example.*

The approach will be illustrated using various examples of a multi-dimensional proposal and target distribution with different correlation levels, and a robust optimization problem.

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