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Accelerated Schwarz Method for Stochastic Elliptic PDEs

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ABSTRACT

This work concerns the resolution of stochastic elliptic equations of the form

$$\nabla \cdot [\kappa(x, \theta) \nabla u(x, \theta)] = -f(x), \quad x \in \Omega, \quad u = u_{\partial\Omega} \text{ on } \partial\Omega, \quad (1)$$

where $\kappa(x, \theta)$ is a random field. We are interested in random fields that cannot be accurately approximated with truncated series, and consider Monte Carlo approaches. In MC approaches, one relies on the multiple resolutions of (1) for different realizations of κ to estimate the statistic of quantities of interest. As millions of samples may be needed to obtain well-converged statistics, efficient solvers are required. We consider a domain decomposition method with a partition of Ω into overlapping subdomains. The additive Schwarz Method (SM) is a fixed-point procedure, where the set of sub-domains' boundary values u_Γ is updated through the parallel resolution of local elliptic problems. The convergence rate of the fixed-point iterations $u_\Gamma^{k+1} = S(u_\Gamma^k | \kappa)$ reduces with the subdomains size and their overlapping. The convergence can be improved using an affine preconditioner A , resulting in the Accelerated SM (ASM) with iterations: $u_\Gamma^{k+1} = S(A(u_\Gamma^k) | \kappa)$.

The exact preconditioner A^{ex} , yielding the fixed-point solution in just one iteration, depends on κ and its computation for a single deterministic problem is usually too expensive. Allocating resources to the construction of an effective κ -dependent preconditioners may, however, be interesting in the Monte Carlo context (see for instance [1]). We explore this idea for preconditioners based on low dimensional approximations of the random field involving N random variables: $\kappa(x, \theta) \approx \kappa(x, \xi)$ where $\xi = (\xi_1, \dots, \xi_N)$. The dependencies of the preconditioner $A(\cdot | \xi)$ on the random variables are subsequently approximated using Polynomial Chaos expansions [2]: $A(\cdot | \xi) \approx A^{PC}(\cdot | \xi) = \sum_\alpha A_\alpha(\cdot) \Psi_\alpha(\xi)$. The expansion coefficients A_α can be determined via a Pseudo-Spectral Projection method [3] requiring the computation of N_p preconditioners, at a cost which is subsequently factorized over the simulated MC samples.

N	2	10	50	100
RMSE $\ \kappa - \hat{\kappa}^N\ $	$3.32 \cdot 10^{-1}$	$1.62 \cdot 10^{-1}$	$7.17 \cdot 10^{-2}$	$5.04 \cdot 10^{-2}$
RMSE $\ A^{ex} - A\ _F$	$1.72 \cdot 10^{-2}$	$1.44 \cdot 10^{-2}$	$5.34 \cdot 10^{-3}$	$2.92 \cdot 10^{-3}$
RMSE $\ A - A^{PC}\ _F$	$2.83 \cdot 10^{-4}$	$1.22 \cdot 10^{-3}$	$2.12 \cdot 10^{-3}$	$2.18 \cdot 10^{-3}$
Size of PC basis (# of PSP nodes)	8 (17)	76 (241)	1,376 (5,201)	5,251 (20,401)
# ASM iterations for $A(\cdot \xi)$	18.9 ± 5.9	14.6 ± 2.9	8.2 ± 0.9	6.8 ± 0.6
# ASM iterations for $A^{PC}(\cdot \xi)$	18.9 ± 5.9	14.7 ± 2.9	8.6 ± 0.9	7.4 ± 0.8

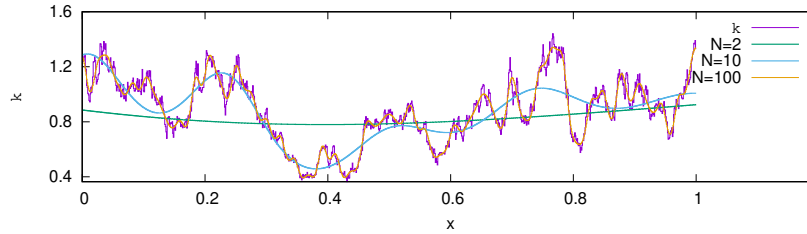


Table 1: Top: Analysis of performance and complexity of the proposed PC-Accelerated Schwarz method (PC-ASM). Below: Approximations for different N of a realization of κ .

We demonstrate the PC-ASM approach on a one-dimension problem with domain $\Omega = [0, 1]$ and discretized with $\approx 1,000$ finite-elements. The partition consists of 20 uniform subdomains with overlapping of 5 elements. The field κ is log-normal with distribution $\log \kappa \sim N(0, C)$, where $C(x, x') = \sigma^2 \exp(-|x - x'|/L)$. The results use $L = 0.2$ and $\sigma^2 = 0.25$. The approximation of κ is defined as

$$\kappa(x, \theta) \approx \hat{\kappa}^N(x, \xi(\theta)) = \exp \left[\sum_{i=1}^N \lambda_i^{1/2} \Phi_i(x) \xi_i(\theta) \right], \quad \xi_i(\theta) = \int \Phi_i(x) \log(\kappa(x, \theta)) dx,$$

where the Φ_i are the dominant eigenfunctions of the covariance C . Table 1 shows for different values of N (top row) the root-mean-squared errors on the field $\hat{\kappa}^N - \kappa$ (second row), on the accelerator $A^{ex} - A$ (in Frobenius norm, third row) and the RMS-error of the PC approximation, $A - A^{PC}$ (fourth row). The fifth row reports the PC basis dimension (for third order expansions) and the number of PSP nodes N_p for the projection. Finally, the number of iterations (averaged value \pm standard deviation) to get $|u_T^k - u_T^{k+1}| < 10^{-9}$ is reported in the last two rows for the ASM using A or its PC approximation A^{PC} respectively. For comparison, the Schwartz method without preconditioning requires 14,250 iterations on average to converge. The table indicates that 1) the preconditioning approach is effective, requiring few iterations even for low N , and 2) the PC approximation of the preconditioner has a low impact on the convergence, especially for the smallest N .

REFERENCES

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